Stability of a Model of Inter-bank Lending

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joint work with Jean-Pierre Fouque

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on Risk, Simulation and Related Topics
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Financial crisis 2007 - 2008

08/09/2007 – BNP Paribas limits withdrawals,
03/16/2008 – Bear Stearns acquisition,
09/15/2008 – Lehman Brothers bankruptcy,
09/16/2008 – $ 85 billion loan to AIG,
10/09/2008 – Interest on reserve balances, ...

- Rare events
- Contagious effect (cascading, snowball, herding behavior) to the whole economy.

How can we prepare for a future financial crisis?

Model Inter-bank Lending Market

Interbank lending market is an institution for banks to lend money.

- When funding of a bank is not available enough, the bank borrows in overnight markets for immediate needs.
- If required collateral is too high, it fails.
- Deficits of banks spread among banks along with monetary flow.

How can we model such an interbank lending market?

- Network model, Cascade and contagious process from Epidemics, Engineering and Physics.
- Intensity based models.
- Here we approximate it by a diffusion model with lending preferences.

On $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ let us consider a banking system $X := (X(t) := (X_1(t), \ldots, X_n(t)), 0 \le t < \infty)$ of $n (\ge 2)$ banks. $X_i(t)$: monetary reserve of bank i at time t with SDE

$$X_i(t) = X_i(0) + \int_0^t \Big[\delta_i + \sum_{j=1}^n (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] \mathrm{d}u$$
$$+ \int_0^t \sum_{k=1}^n \sigma_{ik}(X(u)) \sqrt{X_i(u)} \mathrm{d} W_k(u) ;$$
$$i = 1, \ldots, n , \ 0 \le t < \infty .$$

- Here $W := ((W_1(t), \ldots, W_d(t)), 0 \le t < \infty)$ is the standard d-dimensional Brownian motion, δ_i is a nonnegative constant, $\mathrm{x} := (X_1(0), \ldots, X_n(0)) \in [0, \infty)^n$ is an initial reserve and
- $p_{i,j} : [0, \infty)^n \to [0, 1]$ is bounded, α-Hölder continuous on compact sets in $(0, \infty)^n$ for some $\alpha \in (0, 1]$.
- $a(\cdot) := (a_{ij}(\cdot)) = \sum_{k=1}^n (\sigma_{ik}\sigma_{jk})(\cdot)$ is strictly positive definite, α-Hölder continuous on compact sets for some $\alpha \in (0, 1]$.

Proposition In addition to the assumptions assume that there exists a constant $c_0(\alpha, d) > 0$ such that $a(\cdot) := (a_{ij}(\cdot)) = \sum_{k=1}^{n} (\sigma_{ik}\sigma_{jk})(\cdot)$ satisfies

$$\sum_{i \neq k} |a_{ik}(x)| \leq c_0 \min_i a_{ii}(x) ; \quad x \in \partial \mathbb{R}^n_+ ,$$

then the weak solution $(X, W), (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}_{\mathrm{x}})$ exists and is unique in the sense of probability distribution.

Proof is in Bass & Perkins ('03) based on a martingale problem with a perturbation argument.

- Shiga & Shimizu ('80), Cox, Greven & Shiga ('95) consider $\delta_i = 0$, $p_{i,j}, \sigma_{ik}$ are constants yet in the infinite dimensional case.

Simplification

$$X_i(t) = X_i(0) + \int_0^t \Big[\delta_i + \sum_{j=1}^{n} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] d\, u$$
$$+ 2\int_0^t \sqrt{X_i(u)}\, d\, W_i(u); \quad i = 1, \ldots, n,\ 0 \le t < \infty .$$

- Individual drift δ_i. Simpler case $\delta_i := \delta / n \ge 0$.
- If bank j has more reserve than bank i, that is, $X_j(t) > X_i(t)$ at time t, there is a monetary flow from bank j to bank i proportional to the *preference* $p_{i,j}(\cdot)$.
  △ Discontinuous or heterogeneous $p_{i,j}(\cdot)$, rank-based coefficients ?

△ Remove from the system upon default? – We will see later.

$$X_i(t) = X_i(0) + \int_0^t \Big[\frac{\delta}{n} + \sum_{j=1}^{n} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u)) \Big] d\,u$$

$$+ 2 \int_0^t \sqrt{X_i(u)}\, d\, W_i(u) ; \quad i = 1, \ldots, n, \; 0 \le t < \infty .$$

- The random shock $\sqrt{X_i(\cdot)}\, W_i(\cdot)$ with variance proportional to its size.
  △ correlated BM $\sum_{k=1}^{d} \sqrt{X_i(\cdot)} \sigma_{ik}(X(\cdot))\, W_k(\cdot)$ might lead more interesting phenomena
  (Ichiba & Karatzas (2010)).
  △ Interaction and feedback effects with other economic sectors as in real financial crisis ?

When $p_{i,j}(\cdot) = p_{j,i}(\cdot)$ for $1 \le i,j \le n$, we observe for $x \in \mathbb{R}_+$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (x_j - x_i) \cdot p_{i,j}(x) = \sum_{i<j} (x_j - x_i) \cdot p_{i,j}(x) + \sum_{j<i} (x_j - x_i) \cdot p_{i,j}(x) = 0 .$$

The total reserve $\mathfrak{X}(\cdot) := \sum_{i=1}^{n} X_i(\cdot)$ in the system satisfies

$$\mathfrak{X}(t) = \mathfrak{X}(0) + \delta t + 2 \int_0^t \sum_{j=1}^{n} \sqrt{X_j(u)}\, d\, W_j(u); \quad 0 \le t < \infty .$$

By possibly extending the probability space and introducing another Brownian motion $\beta(\cdot)$, we obtain a squared Bessel process of dimension δ:

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Properties of total monetary reserve $\mathfrak{X}(\cdot) = \sum_{i=1}^{n} X_i(\cdot)$

By the property of the squared Bessel process if $\delta \ge 2$ the total reserve $\mathfrak{X}(\cdot)$ never achieves zero:

$$\mathbb{P}_{\mathrm{x}}(\mathfrak{X}(t) > 0\,,\ \text{for all } t \in [0, \infty)) = 1\,;$$

$$\text{If } \delta > 1\,,\ \ \mathbb{P}_{\mathrm{x}}(\limsup_{t \to \infty} \mathfrak{X}(t) = \infty) = 1\,;$$

$$\text{If } \delta = 2\,,\ \ \mathbb{P}_{\mathrm{x}}(\inf_{0 \le s < \infty} \mathfrak{X}(s) = 0) = 1\,; \quad \mathrm{x} \in (0, \infty)^n\,.$$

If $0 < \delta < 2$, the point $\{0\}$ is instantaneously reflecting.
If $\delta = 0$, the total reserve attains zero in a finite time and stops thereafter almost surely.

Proposition. If the lending preferences $\{p_{i,j}(\cdot), 1 \le i,j \le n\}$ satisfy

$$\sup_{x \in [0,\infty)^n} |x_i - x_j| \cdot p_{i,j}(x) < \frac{(2-\delta)n + \delta k}{n(n-1)(n-k)} =: 2c_0 ; \quad 1 \le i,j \le n ,$$

then every bank except less than or equal to k banks is bankrupt together at some time $t \in (0,\infty)$ almost surely for

$$k \in \{k \,|\, (2-\delta)n + \delta k > 0\} \cap \{1, \ldots, n-1\} ,$$

that is, for every choice $(\ell_1, \ldots, \ell_{n-k})$ of $(n-k)$ banks,

$$\mathbb{P}_{\mathrm{x}}(X_{\ell_1}(t) = X_{\ell_2}(t) = \cdots = X_{\ell_{n-k}}(t) = 0 , \text{ for some } t \in (0,\infty)) = 1 ;$$

for $\mathrm{x} \in [0,\infty)^n$.

Proof is based on comparison theorem by IKEDA & WATANABE (1977). The sum $\mathcal{X}_{n-k}(\cdot) = \sum_{i=1}^{n-k} X_{\ell_i}(\cdot)$ is dominated by a squared Bessel process with dimension

$$\delta_1 := \frac{\delta(n-k)}{n} + \sup_{x \in [0,\infty)^n} \Big| \sum_{i=1}^{n-k} \sum_{j=1}^{n} (x_j - x_{\ell_i}) \cdot p_{\ell_i,j}(x) \Big| < 2 .$$

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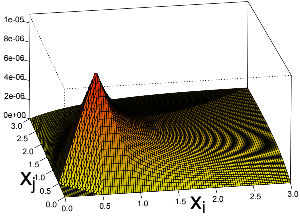
Note that there are possibly many choices of the lending preference that satisfy the above inequality. For example,

$$p_{i,j}(\cdot) \equiv 0; \quad 1 \leq i,j \leq n.$$

Another example is

$$\frac{p_{i,j}(x)}{c_1} = \begin{cases} 2(x_i \wedge x_j)/(x_i + x_j)^2 & \text{if } x_i + x_j \geq 1, \\ 1 - 2(x_i \wedge x_j) & \text{if } x_i \wedge x_j \geq 1/2, 1/2 \leq x_i + x_j \leq 1, \\ 2(x_i + x_j) - 1 & \text{if } x_i \wedge x_j \leq 1/2, 1/2 < x_i + x_j < 1, \\ 0 & \text{otherwise}, \end{cases}$$

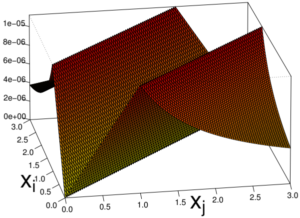
where the constant c_1 is less than c_0.



Similarly, given a nonnegative function $h : [0, \infty) \to [0, 1]$ which is α-Hölder continuous on compact sets in $(0, \infty)$ for some $\alpha \in (0, 1]$, we can take

$$p_{i,j}(x) = h(|x_i - x_j|); \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n, \; 1 \leq i, j \leq n .$$

The condition holds if we choose $c_1 < c_0$ and $h(x) = c_1 / x$ for $x \geq 1$ and $h(x) = c_1 x$ for $x \leq 1$.



Under the same condition, let us consider the default times

$$\tau_{n-k} := \inf\{t \geq 0 : \mathcal{X}_{n-k}(t) = 0\}.$$

By the comparison theorem, we can estimate the tail probability distribution

$$\mathbb{P}_{\mathrm{x}}(\tau_{n-k} > t) \leq \int_t^{\infty} \frac{1}{s\Gamma(\delta_1)}\Big(\frac{a^2}{2s}\Big)^{\delta_1} e^{-\frac{a^2}{2s}}\, d\,s\,; \quad t \geq 0.$$

where $a := \mathcal{X}_{n-k}(0)$ is the initial value of the sum and

$$\delta_1 := \frac{\delta(n-k)}{n} + \sup_{x\in[0,\infty)^n}\Big|\sum_{i=1}^{n-k}\sum_{j=1}^{n}(x_j - x_{\ell_i})\cdot p_{\ell_i,j}(x)\Big| < 2.$$

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Similarly, define

$$\delta_0 := \frac{\delta(n-k)}{n} + \inf_{x\in[0,\infty)^n} \sum_{i=1}^{n-k}\sum_{j=1}^{n}(x_j - x_i)\cdot p_{\ell_i,j}(x)$$

and we obtain

$$\mathbb{P}_{\mathrm{x}}(\tau_{n-k} \geq t) \geq \int_t^{\infty} \frac{1}{s\Gamma(\delta_0)}\Big(\frac{a^2}{2s}\Big)^{\delta_0} e^{-\frac{a^2}{2s}}\, d\, s\,; \quad t \geq 0\,.$$

Probability that many defaults occur in a given time.

If there is no monetary flow $p_{ij}(\cdot) \equiv 0$,

$$\mathbb{P}_{\mathrm{x}}(\#\text{ defaults before time } t \text{ is } k)$$

$$= \sum_{1 \le \ell_1 < \cdots < \ell_k \le n} \Big[\prod_{j=1}^{k} \Big(1 - I\Gamma(X_{\ell_j}(0); \delta/n) \Big) \Big] \Big[\prod_{i \notin (\ell_1, \ldots, \ell_k)} I\Gamma(X_i(0); \delta/n) \Big]$$

where

$$I\Gamma(a, \delta) := \int_t^\infty \frac{1}{s\Gamma(\delta)} \Big(\frac{a^2}{2s} \Big)^\delta e^{-\frac{a^2}{2s}} \, d\, s; \quad a \in [0, \infty).$$

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Unfortunately, it seems very hard to obtain an explicit theoretical answer, for any given lending preference $p_{ij}(\cdot)$.

Instead, here let us discuss a Monte Carlo scheme on how to compute the small probability,

$$\mathbb{P}_{\mathrm{x}}(\mathfrak{n} = k) = \mathbb{E}[\mathbf{1}_{\{\mathfrak{n}=k\}}]; \quad k = 1, \ldots, n,$$

where

$$\mathfrak{n} := \sum_{i=1}^{n} \mathbf{1}_{\{\min_{0 \le s \le T} X_i(s) \le b\}}$$

for some threshold $b > 0$, following the interacting particle method proposed by Carmona, Fouque & Vestal ('09).

Interacting particle system algorithm [Del Moral and Garnier ('05)]

Intuition: consider a background MC $(\xi_k)_{k\geq 0}$ with transition kernel $K_k(\xi_{k-1},\xi_k)$, and its history $\eta_k := (\xi_0,\ldots,\xi_k)$, $k \geq 0$. Given $f_k : \mathbb{R}^{k+1} \to \mathbb{R}$, define

$$\gamma_k(f_k) = \mathbb{E}\Big(f_k(\eta_k) \cdot \prod_{1\leq \ell < k} G_k(\eta_\ell)\Big)$$

with a multiplicative potential function, and its normalized measure

$$\nu_k(f_k) = \frac{\gamma_k(f_k)}{\gamma_k(1)}\,.$$

Since $\gamma_{k+1}(1) = \gamma_k(G_k) = \nu_k(G_k)\gamma_k(1) = \cdots = \prod_{\ell=1}^{n} \nu_\ell(G_\ell)$,

$$\mathbb{E}(f_k(\eta_k)) = \gamma_k(f_k \prod_{1\leq \ell < k} (G_\ell)^{-1}) = \nu_k(f_k \prod_{1\leq \ell < k} (G_\ell)^{-1}) \prod_{1\leq \ell < n} \nu_\ell(G_\ell).$$

Here we can use a recursion: $\eta_1(\cdot) = K_1(\xi_0,\cdot)$,

$$\nu_k(\cdot) = \int \nu_{k-1}(d\eta_{k-1}) \frac{G_{k-1}(\eta_{k-1})}{\nu_{k-1}(G_{k-1})} K_k(\eta_{k-1},\cdot)\,.$$

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Dividing the time interval $[0, T]$ into L equal subintervals $[(\ell-1)T/L, \ell T/L]$ with $\ell = 1 \ldots, L$, we simulate M random chains

$$\{ Y_\ell^{(j)} = (\widehat{X}^{(j)}(\ell T/L), \widehat{m}^{(j)}(\ell T/L))\}_{1 \le \ell \le L}; \quad j = 1, \ldots, M,$$

where $\widehat{X}^{(j)}(\cdot)$ is the jth simulation of $X(\cdot)$ and $\widehat{m}^{(j)}$ is the jth simulation of the vector $m(\cdot) := (m_1(\cdot), \ldots, m_n(\cdot))$ of the running minimum
$m_i(t) = \min_{0 \le s \le t} X_i(s)$, for $1 \le i, j \le n$, $0 \le t \le T$.

After initializing the chain, for each $\ell = 1, \ldots, L$, repeat the following selection and mutation stages

- (Selection Stage). Sampling M new particles from $\{Y_\ell^{(j)}\}_{1 \le j \le M}$ with Gibbs weights

$$\Big(\prod_{i=1}^{n} \gamma_{i,\ell}^{(j)}\Big) \Big(\sum_{j=1}^{M} \prod_{i=1}^{n} \gamma_{i,\ell}^{(j)}\Big)^{-1} \quad \text{where}$$

$$\gamma_{i,\ell}^{(j)} := \Big[\frac{\min(\widehat{m}_i^{(j)}((\ell-1)\,T/L), \widehat{X}_i^{(j)}(\ell\,T/L))}{\widehat{m}_i^{(j)}((\ell-1)\,T/L)}\Big]^{-\alpha},$$

  for each $j = 1, \ldots, M$ with some $\alpha > 0$.

- (Mutation Stage). Running Euler scheme to get the new value $Y_{\ell+1}^{(j)}$, $j = 1, \ldots, M$, starting from the new particles sampled in the above.

The probability estimate of $\mathbb{P}_{\mathrm{x}}(\mathfrak{n}=k)$ is given by

$$\widehat{\mathbb{P}}_{\mathrm{x}}(\mathfrak{n}=k)=\frac{1}{M}\sum_{j=1}^{M}\Big(\mathbf{1}_{\{\widehat{\mathfrak{n}}^{(j)}=k\}}\prod_{i=1}^{n}\Big[\frac{m_i^{(j)}(T)}{m_i^{(j)}(0)}\Big]^{\alpha}\Big)\cdot\Big[\prod_{\ell=0}^{L-1}\Big(\frac{1}{M}\sum_{a=1}^{M}\prod_{i=1}^{n}\gamma_{i,\ell}^{(a)}\Big)\Big];$$

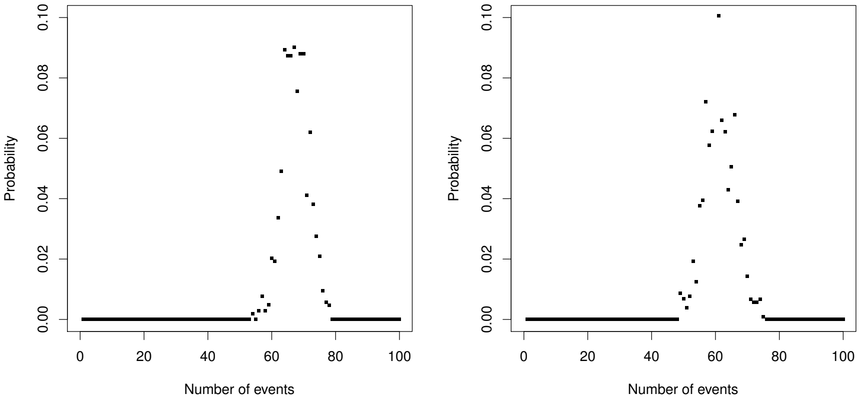
for $k=1,\ldots,n$, where $\widehat{\mathfrak{n}}^{(j)}$ is the corresponding number to $\mathfrak{n}$ in the jth simulation for $j=1,\ldots,M$.

Extreme examples

Set $\mathrm{x} = (1, \ldots, 1)$, $\delta = 2$ and $p_{i,j}(\cdot)$ specified as in the first picture, $T = 1$, $n = 100$, $M = 1000$ (# copies), $L = 10$ (# subintervals of Time), $\alpha = 0.0001$, and run the system with the sub-subinterval for the Euler scheme in the mutation stage as 0.001 to compute

$$\mathbb{P}_{\mathrm{x}}(\mathfrak{n} = k) = \mathbb{P}_{\mathrm{x}}\Big(\sum_{i=1}^{n} \mathbf{1}_{\{\min_{0 \le s \le T} X_i(s) \le b\}} = k\Big); \quad k = 1, 2, \ldots$$

$b = 0.1$ (left), $b = 0.001$ (right).



For error analysis and comparisons with other methods in this context see CARMONA & CRÉPY (2009). □

Stochastic stability

$$X_i(t) = X_i(0) + \int_0^t \Big[\frac{\delta}{n} + \sum_{j=1}^{n} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u)) \Big] d\,u + 2 \int_0^t \sqrt{X_i(u)}\, d\,W_i(u); \quad i = 1, \ldots, n,\ 0 \le t < \infty .$$

When a bank i has the maximum amount of reserve at some time among all the banks, all the outflow from i to j is positive, and it gives the negative pressure to the reserve X_i .

Let us consider the deviation $Y(\cdot) := (Y_1(\cdot), \ldots, Y_n(\cdot))$:

$$Y_i(\cdot) := X_i(\cdot) - \frac{1}{n} \mathfrak{X}(\cdot)$$

from the average $\mathfrak{X}(\cdot)$ for $i = 1, \ldots, n$. Since $\sum_{i=1}^{n} Y_i(\cdot) = 0$,

$Y(\cdot)$ takes values in $\Pi := \{ y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = 0 \}$.

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Proposition[$\delta > 0$ and symmetric $p_{i,j}(\cdot) = p_{j,i}(\cdot)$] If there exist positive constants c_3, c_4 such that lending preference $p_{i,j}(\cdot)$ satisfies a stability condition,

$$\min_{1 \le i,j \le n} \inf_{x \in [0,\infty)^n} \{p_{i,j}(x) : |x_i - x_j| > c_3\} \ge c_4 > 0\,,$$

then the Π-valued process $Y(\cdot)$ is stochastically stable, that is, there exists a probability measure $\mu(\cdot)$ such that the $SLLN$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(Y(t))\, d\, t = \int_\Pi f(y) \mu(dy) \quad a.s.$$

for every bounded measurable function $f : \Pi \to \mathbb{R}$.

Corollary. Under the same condition, the $(n \times n)$ matrix-valued process $(X_i(\cdot) - X_j(\cdot))_{1 \le i,j \le n}$ is stochastically stable.

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Induced random graph

Considering each bank as a node (vertex) and the connection between two banks as a link in the graph. Here we consider the connection between bank i and bank j, in terms of the (absolute) monetary flow $|X_j(\cdot) - X_i(\cdot))| \cdot p_{i,j}(X(\cdot))$ between banks $1 \le i < j \le n$.

For each fixed $r \ge 0$, we *connect* banks i and j with indicator $\chi_{i,j;r}(\cdot) = 1$, if the monetary flow is larger than r, otherwise $\chi_{i,j;r}(\cdot) = 0$, that is,

$$\chi_{i,j;r}(\cdot) := 1_{\{ |X_j(\cdot) - X_i(\cdot))| \cdot p_{i,j}(X(\cdot)) \ge r \}} ; \quad 1 \le i, j \le n .$$

For a directed graph, replace the indicator by the sign

$$\text{sgn} \, [(X_j(\cdot) - X_i(\cdot)) \cdot p_{i,j}(X(\cdot)) - r]; \quad 1 \le i, j \le n .$$

Statistics of graph

- For each i the number of links is called *degree* of bank i: $\mathrm{degree}_{i;r}(\cdot) := \sum_{j=1}^{n} \chi_{i,j;r}(\cdot)$.
- *distance* $\mathrm{dist}_{i,j}(\cdot)$ between i and j is the number of minimum links from i to j.
- *eccentricity* of i is $\max_{1 \le j \le n}[\mathrm{dist}_{i,j}(\cdot)]$, $i = 1, \ldots, n$, and the *diameter* of the network is $\max_{1 \le i < j \le n}[\mathrm{dist}_{i,j}(\cdot)]$.
- *average distance* $\sum_{j=1}^{n} \mathrm{dist}_{i,j}(\cdot) / n$ of bank i also indicates where the bank i is allocated in the network: the bank with smaller average distance is located closer to the center of network.
- influential domain (# of maximal connected nodes in the network), betweenness centrality of bank i, affinity, ... [Müller('06), Soramäki et al ('06), Santos & Cont (2010)]

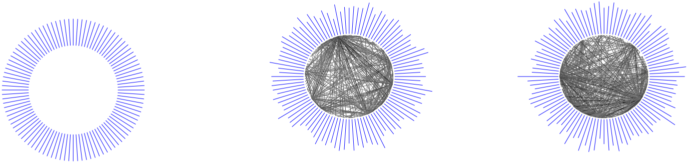


Figure: The graphs of the network for the initial, after 200 steps and 400 steps, respectively.

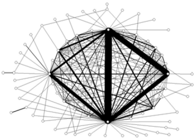
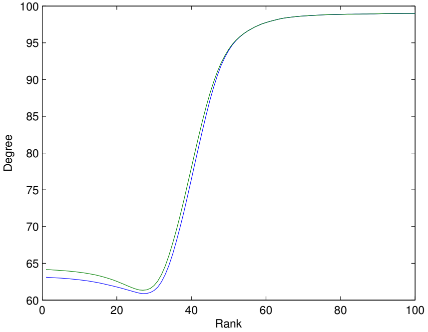


Figure: A snapshot of core Fedwire Interbank Payment Network in 2004 [Soramaki et al. ('06)].

Stability of induced graph

Corollary. Under the stability condition, if every lending preference $x = (x_1, \ldots, x_n) \to p_{i,j}(x)$ depends only on $x_i - x_j$ for $1 \le i, j \le n$, then the monetary flow $F_{i,j}(t) := (X_i(t) - X_j(t)) \cdot p_{i,j}(X(t))$ of the drift coefficient is stochastically stable, and hence so are the statistics (degree, distance, eccentricity, diameter, ...) of the induced graph.



Simulated, expected degrees v.s. the ranking of banks for different r.

Exit system

Default banks are removed from the whole system to the cemetery Δ.

Let us denote the zero sets by $\mathcal{Z}_i := \{x \in [0,\infty)^n : x_i = 0\}$, $i = 1, \ldots, n$, and the initial index set $I_0 := \{i : 1 \le i \le n\}$ with size $|I_0| = n$.
Every time when their monetary reserves become zero, or in one of the zero sets, that is, at the first default time

$$\sigma_1 := \inf\{t > \sigma_0 = 0 : X(t) \in \cup_{i=1}^{n} \mathcal{Z}_i\}$$

we remove all the default banks to Δ and keep the survivors' index $I_{\sigma_1} := \{i : X_i(\sigma_1) \ne 0 \text{ nor } X_i(\sigma_1) \ne \Delta\}$.

For the survived banks $i \in I_{\sigma_1}$ we restart the process with the following SDE:

$$X_i(t) = X_i(\sigma_1) + \int_{\sigma_1}^{t} \Big[\frac{2}{|I_{\sigma_1}|} + \sum_{j \in I_{\sigma_1}} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] d\, u$$

$$+ 2 \int_0^t \sqrt{X_i(u)} d\, W_i(u) ; i \in I_{\sigma_1} , \quad \sigma_1 \le t \le \sigma_2 ,$$

where $X(\cdot) := \{X_i(\cdot) : i \in I_{\sigma_1}\}$ is defined until the next default time $\sigma_2 := \inf\{t > \sigma_1 : X(t) \in \cup_{i \in I_{\sigma_1}} \mathcal{Z}_i\}$.

We continue this exit rule, and build a probability space $(\Omega, \mathcal{F}, \{\mathbb{F}_t\}, \mathbb{P})$ by pasting the probability measure locally at every stopping times $(\sigma_1, \sigma_2, \ldots)$ of defaults.

Let us define the index process $I_t := \{i : X_i(t) \neq 0 \text{ nor } X_i(t) \neq \Delta\}$.

Proposition. If the preference satisfies

$$\sup_{x\in[0,\infty)^n} |x_i - x_j| \cdot p_{i,j}(x) < \frac{(2-\delta)n + \delta k}{n(n-1)(n-k)} =: 2c_0\,; \quad 1 \le i,j \le n\,,$$

then for every $k \in \{k|(2-\delta)n+\delta k > 0\} \cap \{1,\dots,n-1\}$,

$$\mathbb{P}_{\mathrm{x}}(X_{\ell_1}(t) = \cdots = X_{\ell_{n-k}}(t) = 0\,,$$

$$\text{for some } t < \sigma_* \text{ and for some}(\ell_1,\dots,\ell_{n-k})) > 0\,,$$

where $\sigma_* := \inf\{t > 0 : |I_t| \le n-k+1\}$.

Competition or Cooperation?

Example. $[\delta = 2]$ If the preference satisfies

$$\sup_{x \in [0,\infty)^n} |x_i - x_j| \cdot p_{i,j}(x) < \frac{2}{n},$$

then only one bank can survive eventually

$$\lim_{t \to \infty} |I_t| = 1, \quad a.s.$$

for every initial point $\mathrm{x} \in (0, \infty)^n$ and $\inf\{t > 0 : |I_t| = 1\} < \infty$.
On the other hand, if

$$\inf_{x \in [0,\infty)^n} |x_i - x_j| \cdot p_{i,j}(x) > \frac{2(n-1)}{n},$$

then every bank can survive

$$\mathbb{P}_{\mathrm{x}}(|I_t| = n, \quad \text{for all } t > 0) = 1, \quad a.s.; \quad x \in (0, \infty)^n.$$

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Summary.

- Toy model on interbank lending system.
- Risk of multiple defaults.
- Future research:
  - Bridge to Economics literatures - Social welfare "too big to fail"? Lending cost and overnight rate. Relation to the whole economy.
  - Discontinuous or heterogeneous $p_{i,j}(\cdot), \sigma_{ik}(\cdot)$, rank-based coefficients. (c.f. Ichiba et al. (2011))
  - Moment Stability, Lyapunov function, Perturbed system, Concentration-type inequalities (c.f. Ichiba, Pal, Shkolnikov (2011))
  - Control problem for stochastic delay equation. Game theoretic approach, Mean-field type approximation.

Mean-field limit

If the lending preference $(p_{i,j}(\cdot))_{1 \le i,j \le n}$ is strictly positive in the sense that

$$\min_{1 \le i < j \le n} \inf_{x \in (0,\infty)^n} p_{i,j}(x) > 0 \,,$$

then the stability condition is not satisfied. In particular, it contains the case if

$$p_{i,j}(\cdot) = 1 \,/\, n \,; \quad 1 \le i, j \le n \,.$$

As $n \to \infty$, we may consider a mean-field limit:

$$d\, X_i(t) = (\mathbf{m} - X_i(t))\, d\, t + 2\sqrt{X_i(t)}\, d\, W_i(t) \,,$$

where $\mathbf{m}$ is the mean of the monetary reserve distribution.